Markov Chains & Hidden Markov Models

TDT4187 – Exercise 4

Independence of events

1) Consider sample space $\mathcal{S} = \{s_1, s_2, s_3, s_4\}$ with uniform probability distribution over the elementary events.
   Consider events $A = \{s_1, s_2\}$, $B = \{s_1, s_3\}$, $C = \{s_1, s_4\}$.

   (a) Are the events A, B, C independent? Why? [1]

   (b) Define a random variable X on the sample space $\mathcal{S}$ and three events D, E, F such that:

   - Each of D, E, F is a union of at least two equivalence classes on S defined by $\sim$, where $a \sim b$ iff $X(a) = X(b)$.
   - D, E, F are mutually independent and distinct (not necessarily disjoint).

   (c) How many random variables, up to isomorphism (of the discrete probability spaces), exist on this sample space?

2) Consider sample space $\mathcal{S} = \{s_i\}_i$; $|S| = 8$, with uniform probability distribution over the elementary events.

   (a) Define events A, B, C so that, simulatenously:

   - $P(A, B, C) = P(A)P(B)P(C)$
   - A, B, C are **not** *pairwise* independent.

   (b) Define events A, B, C so that they are *mutually* independent.

[1] Intentionally ambiguous.

antonink@idi.ntnu.no

NTNU
Det skapende universitet

Markov chains

3) Consider Markov chain $\mathcal{M}$ with the following transition matrix:

$$P = \begin{pmatrix} \frac{1}{2} & \frac{1}{4} & 0 & \frac{1}{4} & 0 & 0 \\ 0 & 0 & \frac{2}{3} & 0 & 0 & \frac{1}{3} \\ 0 & \frac{2}{3} & 0 & \frac{1}{3} & 0 & 0 \\ 0 & 0 & 0 & \frac{1}{3} & \frac{2}{3} & 0 \\ 0 & 0 & 0 & \frac{1}{2} & \frac{1}{2} & 0 \\ 0 & 0 & 0 & 0 & 0 & 1 \end{pmatrix}$$

(a) Draw the graph representation of this Markov chain.

(b) Given that the Markov chain $\mathcal{M}$ starts at s_1:

- What is the probability that $\mathcal{M}$ is in s_2 (s_4) after 1,2,3,4 steps?
- What is the limiting probability that $\mathcal{M}$ is in s_2 (s_4) as the number of steps goes to ∞?

(c) Does $e_i P^n$ converge to a stationary distribution as $n \to \infty$ for any i?[2] If it does, can you express the limiting distributions for each e_i?

(d) Does $\mathcal{M}$ have a unique stationary distribution? If it doesn't, can you express the space of its stationary distributions?

4) Consider the following Markov chain $\mathcal{M}_n$ parametrized by n:

- There is a 1-on-1 correspondence between states and positions on a $n \times n$ chessboard.
- Transition from state s_1 to state s_2 has a nonzero probability iff the move between corresponding positions is a valid move of a knight (♘). If no move is valid from a position, the corresponding state is *absorbing*.
- Transitions with nonzero probability from each state form a uniform probability distribution.

a) Determine graph and transition matrix representation for $\mathcal{M}_3$.

b) Give a stationary distribution for $\mathcal{M}_3$. Is it unique?

*c) Determine transition matrix for $\mathcal{M}_8$. Can you find its stationary distribution analytically? If not, how would you do it computationally?

*d) Is $\mathcal{M}_8$: (1) Irreducible? (2) Regular? (3) Reversible?

[2] e_i is the unit vector $(\delta_{ij})_j$

Hidden Markov Models

*5) Consider an arbitrary hidden markov model.

- How would you efficiently compute the probability $P(q_t = i, q_{t+1} = j \mid O)$, i.e. the probability that a given state transition occured, given your observed sequences?
- Could you use your findings to determine parameters of the underlying hidden markov model?

6) Consider the weather Markov chain from the lecture (see next page) with temperature measurements. The temperatures are recorded with integer precision. On sunny days, they range from 15 to 21°C, on cloudy days from 13 to 17°C and on rainy days from 10 to 16°C. The probability of the measured temperature is highest in the middle of the range (18, 15, 13°C resp.), and decreases by a factor of two for each degree of difference.

(a) Specify the emission probability matrix B of the Hidden Markov Model described above.

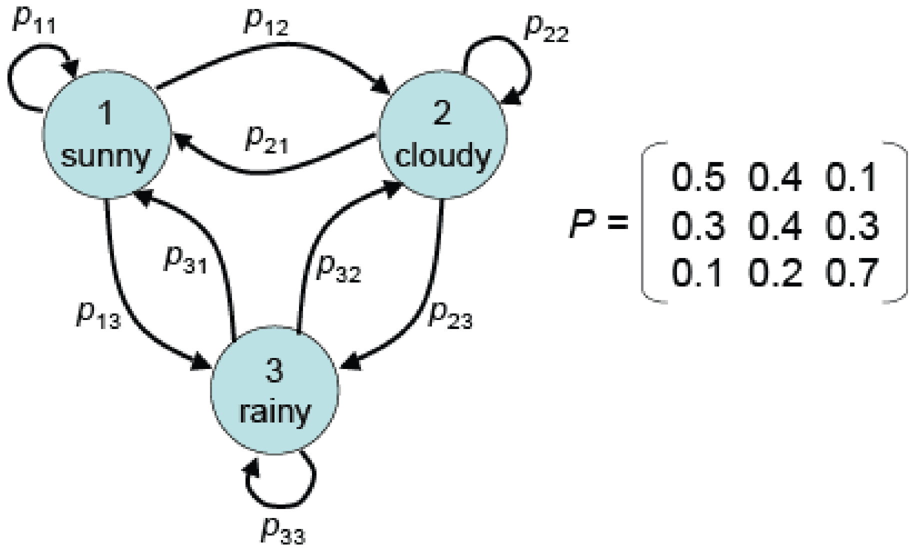
(b) Given that it is sunny on Monday, what is the probability that the week's weather sequence is sunny-cloudy-rainy-cloudy-cloudy-sunny-sunny?

(c) Given that it is sunny on Monday, what is the probability that we observed temperature sequence this week is
17, 15, 11, 16, 17, 21, 17°C ?

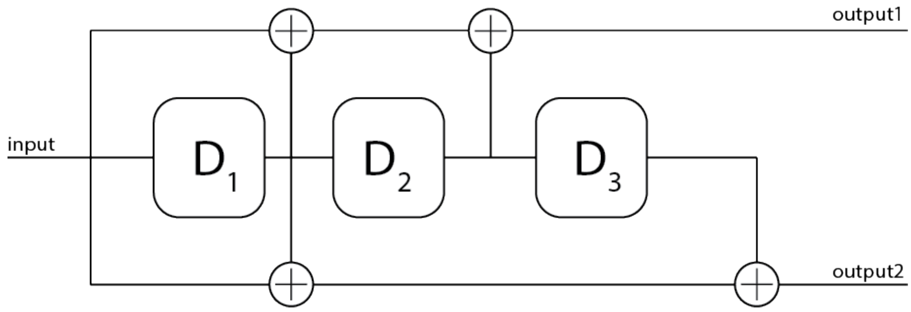
7) Considering the Hidden Markov Model from the previous slide:

(d) Given it is sunny on Monday, what is the probability that this week's temperature sequence fell successively into the following intervals 16-18, 14-16, 10-12, 15-17, 16-18, 20-22, 16-18°C?

(e) What is the most likely weather sequence to have generated the sequence in c)? What is the corresponding probability?



Hidden Markov Models are one of the major models used in coding theory. Consider the convolutional coder described by the following circuit: For



those unfamiliar: There are 2^3 *states of the coder, one for each possible triplet of most recent bits. The most recent bit is in* D_1*, then* D_2*, then* D_3*. The* $\oplus$ *symbol denotes binary* XOR*. The coder generates (2) output bits, which are then sent over a noisy channel and received on the other end. The added redundancy as well as context-specific output then allows for correcting of some transmittion errors on the receiver side.*

*8) For the convolutional coder described:

(a) Consider that the output bits are being sent over a so-called *binary symmetric channel*, with *crossover probability* $\frac{1}{10}$. That is, each bit sent is flipped with chance $\frac{1}{10}$ (independently); otherwise, it is received without error. Create the Hidden Markov model corresponding to the problem, given that the input bits are uniformly independently random.

(b) Use the Viterbi algorithm to identify the most likely encoded sequence given that the received message is 01101110, using the Hidden Markov Model from a). (bit corresponding to output1 preceedes the one corresponding to output2)