1 Transforms and Diff. Eq.

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Laplace

$F(s) = \mathcal{L}(f) = \int_0^\infty e^{-st} f(t)\,dt$

shifting

$\mathcal{L}\{e^{at} f(t)\} = F(s-a)$

$\mathcal{L}\{f(t-a)u(t-a)\} = e^{-as} F(s)$

$\mathcal{L}\{y^{(n)}\} = s^n Y(s) - s^{n-1} y(0) - \ldots - y^{(n-1)}(0)$

$\mathcal{L}\{\int_0^t f(\tau)\,d\tau\} = 1/s\, F(s)$

convolution

$(f * g)(t) = \int_0^t f(\tau) g(t-\tau)\,d\tau$

$\mathcal{L}\{f * g\} = \mathcal{L}(f)\mathcal{L}(g)$

$\int_0^t f(\tau)\,d\tau = 1 * f(t)$

Condition: piecewise continuous and $|f(t)| \leq M e^{kt}$

dirac's delta

$\delta(t-a) = \begin{cases} \infty, & \text{if } t = a \\ 0, & \text{otherwise} \end{cases}$

$\int_0^\infty \delta(t-a) = 1$

$\int_0^\infty g(t)\delta(t-a)\,dt = g(a)$

Fourier

p is period of f if $f(x+p) = f(x)$, $p = 2L$

series

$f(x) = a_0 + \sum_{n=1}^{\infty} \left(a_n \cos\left(\frac{n\pi x}{L}\right) + b_n \sin\left(\frac{n\pi x}{L}\right) \right)$

$a_0 = \frac{1}{2L} \int_{-L}^{L} f(x)\,dx$

$a_n = \frac{1}{L} \int_{-L}^{L} f(x) \cos\left(\frac{n\pi x}{L}\right) dx$

$b_n = \frac{1}{L} \int_{-L}^{L} f(x) \sin\left(\frac{n\pi x}{L}\right) dx$

even: $f(-x) = f(x)$, $b_n = 0$

odd: $f(-x) = -f(x)$, $a_n = 0$

euler

$e^{ix} = \cos(x) + i\sin(x)$

$\cos(nx) = (e^{inx} + e^{-inx})/2$

$\sin(nx) = (e^{inx} - e^{-inx})/(2i)$

complex

$f(x) = \sum_{n=-\infty}^{\infty} c_n e^{\frac{in\pi x}{L}}$

$c_n = \frac{1}{2L} \int_{-L}^{L} f(x) e^{\frac{-in\pi x}{L}} dx$

Parseval

$2a_0^2 + \sum_{n=1}^{\infty} (a_n^2 + b_n^2) = 1/\pi \int_{-\pi}^{\pi} f(x)^2\,dx$

integral

$f(x) = \int_0^\infty A(w)\cos(wx) + B(w)\sin(wx)\,dw$

$A(w) = 1/\pi \int_{-\infty}^{\infty} f(v)\cos(wv)\,dv$

$B(w) = 1/\pi \int_{-\infty}^{\infty} f(v)\sin(wv)\,dv$

transform

$\hat{f}(w) = \mathcal{F}(f) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} f(x) e^{-iwx} dx$

$f(x) = \mathcal{F}^{-1}(\hat{f}) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{\infty} \hat{f}(w) e^{iwx} dw$

$\mathcal{F}\{f'(x)\} = iw\mathcal{F}\{f(x)\}$

$\hat{f}'(w) = \mathcal{F}\{-ixf(x)\}$

convolution

$\mathcal{F}\{f * g\} = \sqrt{2\pi}\,\mathcal{F}(f)\mathcal{F}(g)$

$(f * g)(x) = \int_{-\infty}^{\infty} f(p) g(x-p)\,dp$

DFT

$\hat{f}_n = \sum_{k=0}^{N-1} f_k w^{nk}$, $w = e^{-\frac{2\pi i}{N}}$

$\hat{f} = F_N f$, $f = F_N^{-1} \hat{f} = 1/N \bar{F}_N \hat{f}$, $\underset{\downarrow}{\hat{f}_n}$ $\underset{\downarrow}{\hat{f}_{N-n}}$

$f_k = \cos(2\pi k n/N) \Rightarrow \hat{f} = (0, \ldots, N/2, \ldots, N/2, \ldots, 0)$

$g_k = \cos(2\pi k n/N) \Rightarrow \hat{g} = (0, -iN/2, +iN/2, \ldots 0)$

ODE / PDE

wave: $u_{tt} = c^2 u_{xx}$, heat: $u_t = c^2 u_{xx}$

ODE

$g'' + \lambda^2 g = 0 \Rightarrow g = B\cos(\lambda t) + B^* \sin(\lambda t)$

$g' + \lambda^2 g = 0 \Rightarrow g = B e^{-\lambda^2 t}$

d'Alembert

$u(x,0) = f(x)$, $\partial_t u(x,0) = g(x)$,

$u(x,t) = 1/2[f(x+ct) + f(x-ct)] + 1/2c \int_{x-ct}^{x+ct} g(s)\,ds$

$u(x,t) = \phi(x+ct) + \psi(x-ct)$

sep. av variable

1. $u(x,t) = F(x)G(t)$
   - Sett inn i likning, separer og sett lik k.
   - Sett opp en ODE for F, og en for G.

2. Løs likningen for F. (boundary conditions)
   - $k > 0 \Rightarrow F(x) = C_1 e^{\sqrt{k}x} + C_2 e^{-\sqrt{k}x}$ $\cdots = 0$
   - $k = 0 \Rightarrow F(x) = ax + b$ $\cdots = 0$
   - $k < 0 \Rightarrow F(x) = A\cos(wx) + B\sin(wx) \cdots$
     - $w_n = n\pi/L$
     - $k = -w_n^2$
     - $F_n(x) = \sin(w_n x)$ f.eks.

3. Løs likningen for G, med funnet k
   - Se "ODE", wave bruker 2.orden, heat 1. orden

4. La $u(x,t) = \sum_{n=1}^{\infty} G_n F_n$

5. Bruk initial cond. og Fourier(sinus)rekker
   - Finn B_n av $u(x,0)$
   - Finn B_n^* av $u_t(x,0)$ (for wave)

PDE by Fourier transform

1. Fouriertransformér likningen med hensyn på x.
2. Løs resulterende ODE, uttrykk $\hat{u}(w,t)$
3. Transformér initial cond. og bestem konstant(er)
4. Inverstransformér $\hat{u}(w,t)$ til $u(x,t)$

Partial Deivatives

chain

$\partial_x f(g(x)) = \partial_x f(g(x)) \cdot \partial_x g(x)$

$\partial_t f(\vec{x}(t)) = \vec{\nabla} f(\vec{x}(t)) \cdot \partial_t \vec{x}(t)$

$\vec{\nabla} f = (\partial_{x_1} f, \partial_{x_2} f, \ldots, \partial_{x_n} f)$

$D_{\vec{u}} f(\vec{a}) = \vec{\nabla} f(\vec{a}) \cdot \vec{u}$

$J_{ij}(x) = \partial f_i / \partial x_j$

$H_f = J(\vec{\nabla} f)$

Taylor

$f(\vec{a} + \vec{h}) = f(\vec{a}) + \vec{\nabla} f(\vec{a}) \cdot \vec{h} + \vec{h}^T \cdot H_f(\vec{a}) \cdot \vec{h} + \ldots$

Preliminaries

Taylor

$f(x) = \sum_{k=0}^{m} \frac{(x-a)^k}{k!} f^{(k)}(a) + R_{m+1}(x)$

$f(x+h) = \sum_{k=0}^{m} \frac{h^k}{k!} f^{(k)}(x) + R_{m+1}(x)$

$R_{m+1}(x) = \frac{h^{m+1}}{(m+1)!} f^{(m+1)}(\xi) = O(h^{m+1})$

Intermediate value $f \in C[a,b]$, $x \in [f(a), f(b)] \Rightarrow \exists \xi \in (a,b) : f(\xi) = x$

Mean value $f \in C^1[a,b] \Rightarrow \exists \xi \in (a,b) : f'(\xi) = \frac{f(b)-f(a)}{b-a}$

Mean for integrals $f \in C[a,b]$, $\operatorname{sign}(g(x)) = k$, $x \in [a,b]$

$\Rightarrow \exists \xi \in (a,b) : \int_a^b f(x) g(x)\,dx = f(\xi) \int_a^b g(x)\,dx$

2 Numerics

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Quadratures

simple: $Q[f](a,b) = \sum_{i=0}^{n} w_i f(x_i) \approx I[f](a,b)$

comp.: $I[f](a,b) \approx \sum_{j=0}^{m-1} Q[f](x_j, x_{j+1})$

transform: $[-1,1] \to [a,b]$: $dx = dt(b-a)/2$

$x = t(b-a)/2 + (b+a)/2$

Simpson's:

$S(-1,1) = 1/3\,[f(-1) + 4f(0) + f(1)]$

$S(a,b) = \left(\frac{b-a}{6}\right)[f(a) + 4f\left(\frac{a+b}{2}\right) + f(b)]$

$S_m(a,b) = h/3\,[f(x_0) + 4\sum_{j=0}^{m-1} f(x_{2j+1})$

$+ 2\sum_{j=1}^{m-1} f(x_{2j}) + f(x_{2m})]$

$E_S(a,b) = -(b-a)^5 f^{(4)}(\xi)/2880$

$E_{S_m}(a,b) = -(b-a)h^4 f^{(4)}(\xi)/180$

Trapezoidal:

$T(a,b) = \left(\frac{b-a}{2}\right)[f(a) + f(b)]$

$T_m(a,b) = h\,[\tfrac{1}{2}f(x_0) + \sum_{j=1}^{m-1} f(x_j) + \tfrac{1}{2}f(x_m)]$

$E_T(a,b) = -(b-a)^3 f''(\xi)/12$

$E_{T_m}(a,b) = -(b-a)h^2 f''(\xi)/12$

Midpoint:

$M(a,b) = (b-a)f((a+b)/2)$

$M_m(a,b) = \left(\frac{b-a}{m}\right)\sum_{j=0}^{m-1} f((x_j + x_{j+1})/2)$

$E_M(a,b) = -(b-a)^3 f''(\xi)/24$

$E_{M_m}(a,b) = -(b-a)h^2 f''(\xi)/24$

err. est.:

- $E_m = C_m h^n$, $E_{2m} = C_{2m} h^n$, $C = C_m = C_{2m}$
- Solve $I - Q_m \approx C_m h^n$; $I - Q_{2m} \approx C_{2m} h^n$; for I.

Fixed Point Iterations

The intermediate value theorem proves existence and monotonicity proves uniqueness.

Given g such that $r = g(r)$, and x_0
$x_{k+1} = g(x_k)$, $k = 0, 1, \ldots$

conditions: If g is continuous and $a < g(x) < b$ on $[a,b]$ and $|g'(x)| \le L < 1$

- g has a unique fixed point $r \in (a,b)$
- The iterations converge toward r for $x_0 \in [a,b]$
- The error $e_{k+1} = r - x_{k+1}$ satisfies:
  - $|e_{k+1}| \le L|e_k|$, error reduction rate
  - $|e_{k+1}| \le |x_1 - x_0|\,L^{k+1}/(1-L)$, a-priori est.
  - $|e_{k+1}| \le |x_{k+1} - x_k|\,L/(1-L)$, a-posteriori est.

Newton's method →

$x_{k+1} = x_k - f(x_k)/f'(x_k)$

Assume $f \in C^2$ $I_\delta = [r-\delta, r+\delta]$, if

$|f''(x_1)/f'(x_2)| \le 2M$, for all $x_1, x_2 \in I_\delta$,

⇒ Newton's iterations converge quadratically for x_0 s.t.

$|x_0 - r| \le \min\{1/M, \delta\}$.

← for systems:

$x_{k+1} = x_k - J(x_k)^{-1} f(x_k)$

$A = \begin{bmatrix} a & b \\ c & d \end{bmatrix} \Rightarrow A^{-1} = \frac{1}{ad-bc}\begin{bmatrix} d & -b \\ -c & a \end{bmatrix}$

Numerical PDE

$x_i = ih$, $i = 0..M$, $t_n = nk$, $n = 0..N$

Explicit Euler: forward diff. in t-direction

Stable for wave if $ck/h \le 1$. For heat if $c^2 h/k^2 \le 1/2$

Implicit Euler: backward diff. in t-direction. Unconditionally stable for heat.

Crank Nicolson: Write PDE as $u_t = F$

Then Crank Nicolson method is given by

$\frac{U_i^{n+1} - U_i^n}{k} = \frac{1}{2}(F_i^{n+1} + F_i^n)$

$z = \begin{bmatrix} y \\ y' \\ \vdots \\ y^{(m-1)} \end{bmatrix}$, $z' = \begin{bmatrix} y' \\ y'' \\ \vdots \\ f(t, y, \ldots) \end{bmatrix}$, $z_0 = \begin{bmatrix} y_0 \\ y_0' \\ \vdots \\ y_0^{(m-1)} \end{bmatrix}$

$\begin{array}{c|c} 0 & 0 \\ \hline & 1 \end{array}$ $\qquad$ $\begin{array}{c|cc} \downarrow c_i & a_{ij} \rightarrow & \\ 0 & 0 & 0 \\ 1 & 1 & 0 \\ \hline & 1/2 & 1/2 \end{array}$ $b_i \rightarrow$

Euler's method: $y_{n+1} = y_n + hf(t_n, y_n)$

Heun's method: $k_1 = f(t_n, y_n)$, $k_2 = f(t_n + h, y_n + hk_1)$

$y_{n+1} = y_n + h/2\,(k_1 + k_2)$

Runge-Kutta: $k_i = f(t_n + c_i h, y_n + h\sum_{j=1}^{s} a_{ij} k_j)$

explicit if $a_{ij} = 0$, for $i \ge j$: $y_{n+1} = y_n + h\sum_{i=1}^{s} b_i k_i$

Lipschitz continuous if: $\|f(t,y_1) - f(t,y_2)\| \le L\|y_1 - y_2\|$,

.. for all $t, y_1, y_2 \in D$. If $y' = f(t,y)$ is L.C. ⇒ unique solution in D.

A method is of **order p** if $\|e_N\| = \|y(t_{end}) - y_N\| \le Ch^p$, $h = \frac{t_{end} - t_0}{N}$

Local error estimate: $le_{n+1} = \hat{y}_{n+1} - y_{n+1} \approx l_{n+1}$, where method of y is of order p, and $\hat{y}$ of order $p+1$.

... for Runge-Kutta: $le_{n+1} = h\sum_{i=1}^{s}(\hat{b}_i - b_i)k_i$

$h_{new} = P(Tol/\|le_{n+1}\|)^{\frac{1}{p+1}} h_n$, $P \in [0{,}5, 0{,}95]$

Move forwards if $\|le_{n+1}\| < Tol$

Linear stability: $y' = \lambda y$, $y(0) = y_0$

⇒ $y_{n+1} = R(z)y_n$, $z = \lambda h$

Stability region $S = \{z \in \mathbb{C} : |R(z)| \le 1\}$

For stability, choose h such that $z = \lambda h \in S$

A-stable if S covers $\mathbb{C}^-$. Stable independent of h.

Explicit methods cannot be A-stable.

Order of convergence

Order of convegence p, $e_{k+1} \le M e_k^p$

$p \approx \log(e_{k+2}/e_{k+1}) / \log(e_{k+1}/e_k)$

h-dependent:

$e(h) = \|X - X(h)\|$, $e(h) \le Mh^p$

$p \approx \log(e(h_{k+1})/e(h_k)) / \log(h_{k+1}/h_k)$

Numerical diff. and BVP

differences:

$f' = \begin{cases} (f(x+h) - f(x))/h - h/2\,f''(\xi) & \text{Forward} \\ (f(x) - f(x-h))/h + h/2\,f''(\xi) & \text{Backward} \\ (f(x+h) - f(x-h))/2h - h^2/6\,f'''(\xi) & \text{Central} \end{cases}$

$f'' = \frac{f(x+h) - 2f(x) + f(x-h)}{h^2} - h^2/12\,f^{(4)}(\xi)$